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Maximum likelihood estimation of a bivariate ordered probit model: implementation and Monte Carlo simulations

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Abstract. We discuss the estimation of a two-equation ordered probit model. We have written a Stata command `bioprobit` that computes full-information maximum likelihood estimates of this model. Using Monte Carlo simulations, we compare the performance of this and other estimators under various conditions.

Keywords: st0001, bivariate ordered probit, maximum likelihood, monte carlo simulations

1 Introduction

The ordered univariate probability models have been applied extensively in biostatics, economics, political science and sociology. Estimations of the joint probability distribution of two ordered categorical variables are less common in the literature. The bivariate ordered probit models could be treated as an extension of a standard bivariate probit model when the number of categories of the dependent variables is greater than two. While sharing many properties of bivariate probit estimator, the likelihood function for bivariate ordered probit requires modifications in cases when one of underlying DGP contains an endogenous regressor. Despite a potentially wide applicability of this estimator we know of no general routine for estimation of the bivariate ordered profitability models, both in seemingly unrelated and simultaneous specifications[1]. In this paper we describe the implementation of the general full-information maximum (FIML) algorithm to estimate such models. We compare properties of this and other estimators using Monte Carlo simulations.

The econometric problems of estimating a bivariate ordered probability models arise in a variety of settings. For example:

- Calhoun (1989) studies the problem of censoring of desired family size by the number of children ever born. Both models, with and without censoring is effectively seemingly unrelated ordered probit model for two equations.

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[1] Calhoun (1986) develops a FORTRAN-based routine for the estimation of the seemingly unrelated ordered probit models. Adams (2006) offers a Stata program that implements a FIML algorithm for estimating the SURE bivariate probit models with both dependent variables having three categories

- Lawrence and Palmer (2002) examine the connections between attitudes towards a political actor (Hillary Clinton) and an issue that the actor had been actively involved in promoting in studying public opinion about government-run insurance system. The authors estimate seemingly unrelated bivariate ordered probit model for both dependent variables limited to three categories.
- Adams (2006) analyzes the role of R&D spillovers in shaping industrial allocation between learning and internal research. Two dependent variables represent learning shares for academia and industry and are limited to 3 categories. Adams also estimates standard bivariate ordered probit model using `ml lf` method.
- Scott and Axhausen (2006) use bivariate ordered probit methodology to model the household-level decisions to acquire specific types and numbers of mobility tools. They estimate relationship between number of seasonal tickets and number of cars.

2 Methods

2.1 Model specification

Similar to the univariate ordered probability models, bivariate ordered probability models could be derived from the a latent variable model. Assume that two latent variables y_1^* and y_2^* are determined by:

$$\begin{aligned} y_{1i}^* &= \mathbf{x}_{1i}'\beta_1 + \varepsilon_{1i} \qquad (1) \\ y_{2i}^* &= \mathbf{x}_{2i}'\beta_2 + \gamma y_{1i}^* + \varepsilon_{2i} \qquad (2) \end{aligned}$$

where β_1 and β_2 are vectors of unknown parameters, γ is an unknown scalar, ε_1 and ε_2 are the error terms, and subscript i denotes an individual observation. The explanatory variables in the model satisfy the conditions of exogeneity such that $E(\mathbf{x}_{1i}\varepsilon_{1i}) = 0$ and $E(\mathbf{x}_{2i}\varepsilon_{2i}) = 0$.

We observe two categorical variables y_1 and y_2 such that

$$y_{1i} = \begin{cases} 1 & \text{if } y_{1i}^* \le c_{11} \\ 2 & \text{if } c_{11} < y_{1i}^* \le c_{12} \\ \vdots & \\ J & \text{if } c_{1J-1} < y_{1i}^* \end{cases} \qquad y_{2i} = \begin{cases} 1 & \text{if } < y_{2i}^* \le c_{21} \\ 2 & \text{if } c_{21} < y_{2i}^* \le c_{22} \\ \vdots & \\ K & \text{if } c_{2K-1} < y_{2i}^* \end{cases} \qquad (3)$$

The unknown cutoffs satisfy the condition that $c_{11} < c_{12} < \cdots < c_{1J-1}$ and $c_{21} < c_{22} < \cdots < c_{2K-1}$. We define $c_{10} = c_{20} = -\infty$ and $c_{1J} = c_{2K} = \infty$ in order to avoid handling the boundary cases separately.

The probability that $y_{1i} = j$ and $y_{2i} = k$ is:

$$\begin{aligned} Pr(y_{1i} = j,\, y_{2i} = k) &= Pr(c_{1j-1} < y_{1i}^* \le c_{1j},\, c_{2k-1} < y_{2i}^* \le c_{2k}) \\ &= Pr(y_{1i}^* \le c_{1j},\, y_{2i}^* \le c_{2k}) \end{aligned}$$

$$
\begin{aligned}
&- \quad Pr(y^*_{1i} \le c_{1j-1},\, y^*_{2i} \le c_{2k})\\
&- \quad Pr(y^*_{1i} \le c_{1j},\, y^*_{2i} \le c_{2k-1})\\
&+ \quad Pr(y^*_{1i} \le c_{1j-1},\, y^*_{2i} \le c_{2k-1}) \qquad (4)
\end{aligned}
$$

If ε_{i1} and ε_{i2} are distributed as bivariate standard normal with correlation ρ the individual contribution to the likelihood function could be expressed as:

$$
\begin{aligned}
Pr(y_{1i} = j,\, y_{2i} = k) \;&=\; \Phi_2(c_{1j} - \mathbf{x}'_{1i}\beta_1,\, (c_{2k} - \gamma\mathbf{x}'_{1i}\beta_1 - \mathbf{x}'_{2i}\beta_2)\zeta,\, \tilde{\rho})\\
&-\; \Phi_2(c_{1j-1} - \mathbf{x}'_{1i}\beta_1,\, (c_{2k} - \gamma\mathbf{x}'_{1i}\beta_1 - \mathbf{x}'_{2i}\beta_2)\zeta,\, \tilde{\rho})\\
&-\; \Phi_2(c_{1j} - \mathbf{x}'_{1i}\beta_1,\, (c_{2k-1} - \gamma\mathbf{x}'_{1i}\beta_1 - \mathbf{x}'_{2i}\beta_2)\zeta,\, \tilde{\rho})\\
&+\; \Phi_2(c_{1j-1} - \mathbf{x}'_{1i}\beta_1,\, (c_{2k-1} - \gamma\mathbf{x}'_{1i}\beta_1 - \mathbf{x}'_{2i}\beta_2)\zeta,\, \tilde{\rho}) \qquad (5)
\end{aligned}
$$

where Φ_2 is the bivariate standard normal cumulative distribution function, $\zeta = \frac{1}{\sqrt{1+2\gamma\rho+\gamma^2}}$ and $\tilde{\rho} = \zeta(\gamma+\rho)$. We refer to this specification as simultaneous bivariate ordered probit model. If $\gamma = 0$ the model simplifies in such a way that $\zeta = 1$ and $\tilde{\rho} = \rho$. This is a seemingly unrelated specification.

The logarithmic likelihood of an observation i is then

$$
\ln L_i = \sum_{j=1}^{J}\sum_{k=1}^{K} I(y_{1i} = j,\, y_{2i} = k) \ln Pr(y_{1i} = j,\, y_{2i} = k) \qquad (6)
$$

Under assumptions that observations are independent we can sum (6) across observations to get the log likelihood for the entire sample of size N:

$$
\ln \mathcal{L} = \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{k=1}^{K} I(y_{1i} = j,\, y_{2i} = k) \ln Pr(y_{1i} = j,\, y_{2i} = k) \qquad (7)
$$

2.2 Identification

The parameters in the system of equations (1)-(3) are identified only by imposing an exclusion restriction on vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ i.e. at least one element of $\mathbf{x}_1$ should not be present in $\mathbf{x}_2$. We cannot use the nonlinearity as a source of identification as it is done in many limited dependent variable setups (e.g. Heckman model) because in the case of exclusion restriction failure, i.e. if $\mathbf{x}_1 = \mathbf{x}_2$, the linear system (1-2) becomes

$$
\begin{aligned}
y^*_{1i} &= \mathbf{x}'_{1i}\beta_1 + \varepsilon_{1i} \qquad (8)\\
y^*_{2i} &= \mathbf{x}'_{1i}(\beta_2 + \gamma\beta_1) + \varepsilon_{2i} + \gamma\varepsilon_{1i} \qquad (9)
\end{aligned}
$$

The system 8is unidentified. If we can find some variables that are believed to be correlated with y^*_1 but are independent of the ε_1, these variables could be included in $\mathbf{x}_1$ to obtain the consistent estimates of γ, β_2, and ρ.

3 Monte Carlo simulations

The derivation of the exact asymptotic properties of bivariate ordered probability models with endogenous right-hand side variable could be extremely complicated and perhaps not feasible at all. Even if such formulae would exist they provide no insight about the small sample properties of these estimators. In this section we present an empirical evidence about the behavior of our seemingly unrelated and simultaneous `bioprobit` estimator using the Monte Carlo experiments.

3.1 Data-generating process

We generate x_1, x_2 and z as independent standard normal random variables. Shocks e_1 and e_2 have a correlation ρ (we will try different assumptions about the distribution functions). The latent variables y_1^* and y_2^* are

$$\begin{aligned} y_{1i}^* &= \beta_{10} + \beta_{11}x_{1i} + \beta_{12}x_{2i} + \beta_{13}z_i + e_{1i} \\ y_{2i}^* &= \gamma y_{1i}^* + \beta_{21}x_{1i} - \beta_{22}x_{2i} + e_{2i} \end{aligned} \tag{10}$$

The observed dependent variables y_1 and y_2 are defined as:

$$y_{1i} = \begin{cases} 1 & \text{if } \quad y_{1i}^* \le -7 \\ 2 & \text{if } -7 < y_{1i}^* \le -1 \\ 3 & \text{if } -1 < y_{1i}^* \le 0 \\ 4 & \text{if } \quad 0 < y_{1i}^* \le 3 \\ 5 & \text{if } \quad 3 < y_{1i}^* \end{cases} \qquad y_{2i} = \begin{cases} 1 & \text{if } \quad y_{2i}^* \le -7 \\ 2 & \text{if } -7 < y_{2i}^* \le -2 \\ 3 & \text{if } -2 < y_{2i}^* \le -1 \\ 4 & \text{if } -1 < y_{2i}^* \le 1 \\ 5 & \text{if } \quad 1 < y_{2i}^* \le 2 \\ 6 & \text{if } \quad 2 < y_{2i}^* \end{cases} \tag{11}$$

We estimate simultaneous specification in both cases of correct and incorrect distributional assumptions about the error terms; then we check efficiency of our SUR estimator. We compare performance of the FIML estimator with three possible alternatives:

- we estimate system (10) with 2SLS;
- we estimate each equation separately using univariate ordered probit method. In case of simultaneity ($\gamma \neq 0$), y_1 is plugged as an explanatory variable into the equation for y_2 in (10)-(11). We refer to these estimates as independent ordered probit (IOP) method in our simulations;
- we apply a "two-step" procedure, in which we estimate the first equation of system 10 by univariate ordered probit, predict y_{1i}^* based on the estimated parameters, and use this predicted variable as a regressor in the second univariate ordered probit estimation.

For all simulations, values for the parameters in (10) are $\beta_{10} = \beta_{11} = \beta_{13} = \beta_{21} = 1$, $\beta_{12} = 2$, and $\beta_{22} = -2$. In the simultaneous model $\gamma = 0.4$.

3.2 Simultaneous model with normally distributed error terms

In this simulation, error terms e_1 and e_2 are distributed as bivariate standard normal with correlation ρ. We run 1000 replications for each value of $\rho = \{0, 0.5, 0.9\}$ and for the number of observations ranging from 50 to 15000. We report sample means and sample standard deviations of these 1000 estimates. Table 1 shows the results of simulations for the parameter γ in (10) estimated by FIML and three alternative estimators for different values of ρ and different sample size.

For the small samples, the FIML estimator performs well recovering the true parameter $\gamma = 0.4$. However the precision of this estimator is quite low for the sample sizes less than 300 observations. On samples of 3000 observations and more FIML produces unbiased estimates of γ with small variance. At the same time, the estimates of γ generated by any of the three alternative methods are biased even for the sample sizes of 15000 observations. The bias in estimating γ increases for higher ρ in both, IOP and two-step methods. The 2SLS method also produces biased estimates of γ but the size of the bias depends neither on ρ nor on the sample size.

These results are as expected, because both, 2SLS and two-step methods assume (incorrectly) that values of categorical variable y_1 approximate those of the latent continuous variable y_1^*: in estimating the equation for y_2^* in (10) 2SLS uses $\mathbf{x}_1 \left(\mathbf{x}_1'\mathbf{x}_1\right)^{-1} \mathbf{x}_1'\mathbf{y}_1$ (the predicted value from the first-stage OLS) and IOP method uses just y_1, which cannot in general be expected to carry distributional information about the unobserved y_1^* (except of course, ordering of the categories). While the two-step method comes somehow close in estimating the true value of γ when $\rho = 0$, its bias increases when the interaction between two equations in (10) gets stronger when $\rho \neq 0$.

In addition, FIML provides consistent estimate for ρ (not shown here) and also correct standard errors for the parameters; non of the alternative methods do so.

3.3 Simultaneous model: non-normal shocks

In Table 2 we present the results of M-C simulations for DGP where we modify the errors e_1 and e_2 to be non-normal by altering the skewness and kurtosis.

We apply following transformation to generate the skewness in the errors:

$$\begin{aligned} e_1' &= \frac{\chi^2_{(5,F(e_1))}}{\sqrt{10}} \\ e_2' &= \frac{\chi^2_{(5,F(e_2))}}{\sqrt{10}} \end{aligned}$$

and

$$\begin{aligned} e_1' &= \frac{t_{(4,F(e_1))}}{\sqrt{2}} \\ e_2' &= \frac{t_{(4,F(e_2))}}{\sqrt{2}} \end{aligned}$$

in order to get the kurtosis excess.

We generate 1000 replications for $\rho = 0.3$. We do not show the results for 2SLS and

Table 1: Performance of estimators for models with normaly distributed error terms

	# obs. per trial	$\rho =$					
		0.0		0.5		0.9	
		mean	(s.d.)	mean	(s.d.)	mean	(s.d.)
FIML	50	0.413	(0.669)	0.434	(0.385)	0.431	(0.339)
2SLS		0.628	(0.374)	0.602	(0.351)	0.583	(0.357)
IOP		0.671	(0.290)	1.065	(0.325)	1.425	(0.364)
two-step		0.352	(0.209)	0.296	(0.173)	0.262	(0.159)
FIML	100	0.395	(0.148)	0.410	(0.180)	0.414	(0.205)
2SLS		0.616	(0.224)	0.596	(0.224)	0.580	(0.229)
IOP		0.631	(0.178)	1.006	(0.197)	1.352	(0.227)
two-step		0.367	(0.138)	0.312	(0.118)	0.276	(0.104)
FIML	300	0.403	(0.079)	0.408	(0.096)	0.405	(0.104)
2SLS		0.617	(0.122)	0.605	(0.121)	0.594	(0.121)
IOP		0.625	(0.093)	0.978	(0.104)	1.292	(0.122)
two-step		0.373	(0.074)	0.320	(0.065)	0.288	(0.058)
FIML	500	0.400	(0.062)	0.402	(0.074)	0.406	(0.080)
2SLS		0.619	(0.098)	0.600	(0.095)	0.589	(0.095)
IOP		0.617	(0.080)	0.962	(0.085)	1.280	(0.095)
two-step		0.372	(0.056)	0.318	(0.048)	0.289	(0.045)
FIML	1000	0.397	(0.041)	0.397	(0.049)	0.404	(0.056)
2SLS		0.613	(0.065)	0.601	(0.067)	0.590	(0.066)
IOP		0.612	(0.051)	0.958	(0.058)	1.273	(0.066)
two-step		0.370	(0.039)	0.320	(0.035)	0.291	(0.032)
FIML	3000	0.401	(0.024)	0.400	(0.029)	0.400	(0.033)
2SLS		0.616	(0.038)	0.603	(0.038)	0.591	(0.040)
IOP		0.615	(0.030)	0.961	(0.035)	1.273	(0.038)
two-step		0.372	(0.023)	0.321	(0.019)	0.291	(0.019)
FIML	5000	0.400	(0.019)	0.400	(0.023)	0.396	(0.024)
2SLS		0.614	(0.029)	0.598	(0.030)	0.589	(0.030)
IOP		0.613	(0.023)	0.960	(0.026)	1.269	(0.031)
two-step		0.371	(0.017)	0.318	(0.015)	0.290	(0.015)
FIML	10000	0.401	(0.014)	0.401	(0.016)	0.400	(0.017)
2SLS		0.613	(0.021)	0.602	(0.021)	0.593	(0.022)
IOP		0.613	(0.017)	0.958	(0.019)	1.271	(0.020)
two-step		0.371	(0.012)	0.321	(0.011)	0.292	(0.010)
FIML	15000	0.400	(0.011)	0.399	(0.013)	0.400	(0.014)
2SLS		0.614	(0.017)	0.602	(0.017)	0.594	(0.017)
IOP		0.612	(0.013)	0.956	(0.016)	1.271	(0.017)
two-step		0.371	(0.010)	0.320	(0.009)	0.293	(0.008)

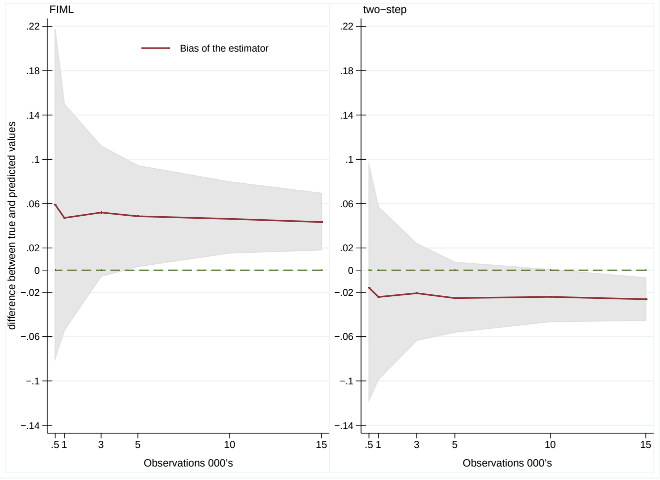
IOP estimators, because similar to the normal case, these estimators fail to produce a reasonable approximation for the true γ.

Table 2: Performance of estimators for the DGP with non-normal shocks

	# obs. per trial	skewness mean	(s.d.)	95% conf. coverage	kurtosis mean	(s.d.)	95% conf. coverage
FIML	500	0.460	(0.091)	92.7%	0.409	(0.073)	94.5%
two-step		0.382	(0.064)	93.2%	0.352	(0.055)	81.9%
FIML	1000	0.446	(0.063)	90.0%	0.415	(0.051)	92.7%
two-step		0.376	(0.048)	87.3%	0.350	(0.039)	70.9%
FIML	3000	0.451	(0.035)	70.1%	0.409	(0.028)	92.8%
two-step		0.379	(0.026)	84.3%	0.349	(0.022)	39.9%
FIML	5000	0.448	(0.028)	55.9%	0.411	(0.022)	78.1%
two-step		0.374	(0.020)	72.5%	0.349	(0.017)	16.4%
FIML	10000	0.447	(0.019)	31.3%	0.411	(0.016)	75.5%
two-step		0.376	(0.015)	56.7%	0.351	(0.012)	2.3%
FIML	15000	0.443	(0.016)	19.9%	0.416	(0.013)	31.1%
two-step		0.374	(0.012)	37.8%	0.354	(0.009)	0.1%

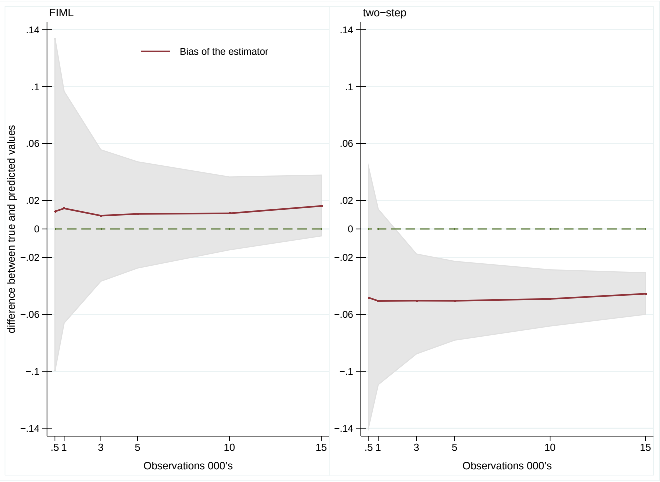
When the error terms are distributed non-normally, FIML produces biased results. The bias is larger for skewed distributions and the size of the bias is independent from the sample size. These results should not be surprising since FIML heavily depends on normality assumption and in case of non-normal shocks estimation method is seriously flowed. FIML produces less biased results then the two-step method in case of the shocks with modified kurtosis. The performance of two methods is similar for the skewed error terms. But one should employ caution interpreting coverage rates for the two-step

Figure 1: Bias of $\hat{\gamma}$: Non-normal shocks with skewness, $\rho = 0.3$



estimator, because estimated standard errors of $\hat{\gamma}$ are incorrect, since $\mathbf{x}_1'\hat{\beta}_1$ has an error not taken into account while estimating the equation for y_2[2].

Figure 2: Bias of $\hat{\gamma}$: non-normal shocks with kurtosis, $\rho = 0.3$



Figures 1 and 2 show the biases and the 95% confidence intervals in estimating the true value of γ for FIML and two-step method by number of observations.

3.4 Seemingly unrelated specification ($\gamma = 0$)

In Table 3 we present estimates for β_{12} (the true value $= 2$) in, what we refer as, the seemingly unrelated bivariate ordered probit specification. The two-step and IOP methods become identical so we refer to these alternatives as two-step; 2SLS reduces to two independent OLS regressions.

As in the simultaneous setup, 2SLS fails to provide a reasonable approximation to the true value of the parameter. It produces downward biased estimates regardless of number of observations.

When the two equations in (10) are independent($\rho = 0$), our FIML estimator and the two-step method yield the same results, both in terms of small-sample biases and efficiency.

The two estimates start to diverge when $|\rho|$ gets higher. The FIML performs better for the small sample sizes ($N < 500$). By construction, the results of both, two-step and 2SLS estimators are independent of ρ. As it can be seen from the table 4, similar to the simultaneous case, introduction of non-normal errors leads to biases in all estimations, with a little less severe effect for modified kurtosis then skewness.

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[2] deriving correct standard errors for the two-step estimator was beyond the scope of this paper

Table 3: Performance of estimators for SUR models with normaly distributed error terms

	# obs. per trial	$\rho =$ 0.0		$\rho =$ 0.5		$\rho =$ 0.9	
		mean	(s.d.)	mean	(s.d.)	mean	(s.d.)
FIML	50	2.347	(0.573)	2.320	(0.547)	2.228	(0.462)
two-step		2.346	(0.567)	2.346	(0.567)	2.346	(0.567)
2SLS		0.745	(0.087)	0.745	(0.087)	0.745	(0.087)
FIML	100	2.142	(0.288)	2.130	(0.280)	2.106	(0.254)
two-step		2.141	(0.287)	2.141	(0.287)	2.141	(0.287)
2SLS		0.738	(0.056)	0.738	(0.056)	0.738	(0.056)
FIML	300	2.037	(0.134)	2.026	(0.132)	2.027	(0.121)
two-step		2.036	(0.134)	2.036	(0.134)	2.036	(0.134)
2SLS		0.736	(0.032)	0.736	(0.032)	0.736	(0.032)
FIML	500	2.025	(0.102)	2.025	(0.101)	2.018	(0.095)
two-step		2.025	(0.102)	2.025	(0.102)	2.025	(0.102)
2SLS		0.737	(0.026)	0.737	(0.026)	0.737	(0.026)
FIML	1000	2.014	(0.075)	2.015	(0.074)	2.010	(0.071)
two-step		2.014	(0.075)	2.014	(0.075)	2.014	(0.075)
2SLS		0.738	(0.018)	0.738	(0.018)	0.738	(0.018)
FIML	3000	2.006	(0.041)	2.006	(0.041)	2.006	(0.038)
two-step		2.006	(0.041)	2.006	(0.041)	2.006	(0.041)
2SLS		0.737	(0.010)	0.737	(0.010)	0.737	(0.010)
FIML	5000	2.006	(0.034)	2.004	(0.034)	2.006	(0.031)
two-step		2.006	(0.034)	2.006	(0.034)	2.006	(0.034)
2SLS		0.737	(0.008)	0.737	(0.008)	0.737	(0.008)
FIML	10000	2.002	(0.023)	2.003	(0.023)	2.003	(0.021)
two-step		2.002	(0.023)	2.002	(0.023)	2.002	(0.023)
2SLS		0.738	(0.006)	0.738	(0.006)	0.738	(0.006)
FIML	15000	2.001	(0.018)	2.001	(0.018)	2.001	(0.017)
two-step		2.001	(0.018)	2.001	(0.018)	2.001	(0.018)
2SLS		0.737	(0.005)	0.737	(0.005)	0.737	(0.005)

In the continuous setup, with correlated error terms, one would get more efficient results from estimating two-equation system simultaneously rather then equation by equation. However, in the case of limited dependent variables, that extra information from the correlation structure is lost in the process of grouping. That is why, for the moderate to large sample sizes (3000-15000), estimating each equation separately using univariate ordered probit model gives results as good as FIML, even if the $|\rho|$ is high.

Table 4: SUR models with the modified error terms

	# obs. per trial	$\rho = 0.5$			
		Skewness		kurtosis	
		mean	(s.d.)	mean	(s.d.)
FIML	50	2.500	(0.617)	2.651	(0.600)
two-step		2.541	(0.996)	2.690	(0.621)
2SLS		0.613	(0.093)	0.754	(0.084)
FIML	100	2.361	(0.369)	2.345	(0.300)
two-step		2.381	(0.391)	2.362	(0.310)
2SLS		0.614	(0.059)	0.744	(0.059)
FIML	300	2.189	(0.187)	2.152	(0.170)
two-step		2.203	(0.199)	2.164	(0.171)
2SLS		0.614	(0.036)	0.744	(0.033)
FIML	500	2.155	(0.141)	2.081	(0.132)
two-step		2.161	(0.151)	2.077	(0.134)
2SLS		0.614	(0.027)	0.741	(0.025)
FIML	1000	2.165	(0.100)	2.059	(0.097)
two-step		2.172	(0.101)	2.049	(0.098)
2SLS		0.612	(0.019)	0.740	(0.018)
FIML	3000	2.144	(0.057)	2.051	(0.055)
two-step		2.151	(0.056)	2.048	(0.055)
2SLS		0.610	(0.011)	0.740	(0.010)
FIML	5000	2.144	(0.044)	2.041	(0.040)
two-step		2.152	(0.046)	2.041	(0.040)
2SLS		0.611	(0.009)	0.740	(0.008)

3.5 Summary

When the error terms are distributed as bivariate normal, the FIML method produces unbiased and more efficient estimates compared to the three alternative estimators. The advantage of the FIML is more visible for simultaneous models ($\gamma \neq 0$) or when $|\rho|$ is high. The FIML estimates are biased when the shocks are not normal. The bias persists even in the large samples. The two-step estimator performs better with the distribution of the error terms is skewed. In the seemingly unrelated setup ($\gamma = 0$), the FIML estimator demonstrates no efficiency gains relative to two-step method. It however, performs better on the small samples and especially for higher values of $|\rho|$. The FIML estimator is the only method that estimate the value of ρ.

4 Algorithm Implementation

A full-information maximum-likelihood estimation of the model is implemented as an `ml d2` evaluator that computes the log likelihood for each observation, along with its analytical gradient and Hessian matrix[3]. To make sure that estimated ρ satisfies $-1 \leq \rho \leq 1$ and cutoffs are ordered ascending we transform the parameters before passing them to `ml`. We estimate $\operatorname{arctanh}(\rho)$ in place of ρ, and $(c_{11}, \sqrt{c_{12} - c_{11}}, \ldots, \sqrt{c_{1J-1} - c_{1J-2}})$ and $(c_{21}, \sqrt{c_{22} - c_{21}}, \ldots, \sqrt{c_{2K-1} - c_{2K-2}})$ in place of $(c_{11}, \ldots, c_{1J-1})$ $(c_{21}, \ldots, c_{2K-1})$ respectively. Results are displayed for the original parameters.

The initial values passed to `ml` are obtained by estimating independent ordered probit model for each equation.

5 The `bioprobit` command

5.1 Syntax

The bioprobit command fits maximum-likelihood two equation ordered probit models either in seemingly unrelated or in simultaneous specifications.

`bioprobit` *depvar1 depvar2 varlist* [*if*] [*in*] [*weight*] [`,` `offset1(`*varname*`)` `offset2(`*varname*`)` `collinear` `robust` `cluster(`*varname*`)` `level(`*#*`)` *maximize_options*]

Or

`bioprobit` `(`*depvar1* [`=`] *varlist1*`)` `(`*depvar2* [`=`] *varlist2*`)` [*if*] [*in*] [*weight*] [`,` `endogenous` `offset1(`*varname*`)` `offset2(`*varname*`)` `collinear` `robust` `cluster(`*varname*`)` `level(`*#*`)` *maximize_options*]

`fweight`s, `pweight`s, and `iweight`s are allowed.

5.2 Options

`endogenous` specifies that the model to be estimated is simultaneous ($\gamma \neq 0$). If available, instrument(s) should be included in *varlist1*.

`offset1(`*varname*1`)` `offset2(`*varname*2`)` specify that *varname*1 and/or *varname*2 be included in the model with the coefficient constrained to be 1.

`robust` computes robust estimates of variance.

`cluster(`*varname*`)` adjusts standard errors for intragroup correlation.

`level(`*#*`)` sets the level for confidence intervals, in percent. The default is `level(95)` or as set by `set level`.

maximize_options are passed directly to `ml` and control the maximization process.

---

[3] See appendix for the derivations of first and second derivatives of the log likelihood function

5.3 Postestimation

The following statistics are available for the `predict` postestimation command:

`outcome(j, k)` or `outcome(#j, #k)` computes the predicted probability $Pr(y_{1i} = j,\ y_{2i} = k)$ or $Pr(y_{1i} = y_1^j,\ y_{2i} = y_2^k)$. If one of the arguments is missing then result will be the marginal probability, i.e. `outcome(., k)` will return $Pr(y_{2i} = k) = \sum_{j=1}^{J} Pr(y_{1i} = j,\ y_{2i} = k)$.

`xb1` calculates $\mathbf{x}_{1i}'\hat{\beta}_1$.

`xb2` calculates $\mathbf{x}_{2i}'\hat{\beta}_2$.

`stdp1` calculates the standard error of the linear prediction of equation 1.

`stdp2` calculates the standard error of the linear prediction of equation 2.

6 Example

We illustrate the use of the `bioprobit` command by estimating the economic gradient in self-assessed health status on the survey data for Russia (Lokshin and Ravallion (2005)). The economic model postulates the bivariate relationship between individual's health status and economic welfare. Neither characteristic is directly observable. Respondents report the categories of the subjective assessment of their health and wealth status.

The underlying model consists of two equation relating the latent health (H) and wealth (W) status to individual characteristics of the respondents $\mathbf{x}$:

$$\begin{aligned} W_i &= \mathbf{x}_{1i}'\beta_1 + \varepsilon_{1i} \qquad (12) \\ H_i &= \gamma W_i + \mathbf{x}_{2i}'\beta_2 + \varepsilon_{2i} \qquad (13) \end{aligned}$$

The observed variables for the individual's subjective health (SAH) and subjective wealth (SAW) assessments are related to the corresponding latent variables as:

$$SAH_i = \begin{cases} 1 \text{ - very bad} & \text{if } \quad H_i \le \mu_1 \\ 2 \text{ - bad} & \text{if } \mu_1 < H_i \le \mu_2 \\ 3 \text{ - average} & \text{if } \mu_2 < H_i \le \mu_3 \\ 4 \text{ - good} & \text{if } \mu_3 < H_i \le \mu_4 \\ 5 \text{ - very good} & \text{if } \mu_4 < H_i \end{cases} \qquad SAW_i = \begin{cases} 1 \text{ - poor} & \text{if } \quad W_i \le \delta_1 \\ 2 & \text{if } \delta_1 < W_i \le \delta_2 \\ \vdots & \\ 7 \text{ - rich} & \text{if } \delta_6 < W_i \end{cases}$$

Assuming that ε_1 and ε_2 are distributed normally $N(0, \sum)$ the system could be estimated by FIML implemented in `bioprobit`. The system (12)-(13) is identified by non-linearity (although weakly), but we introduce an instrument in equation (12) to improve identification properties of the model. As an instrument we the logarithm of per-capita consumption of households averaged across their particular locality of residence, The paper argues that this variable while directly affecting the subjective wealth perception has no direct effect on the subjective assessment of health.

```
.  bioprobit (econrnk `ostr' mean_logexp) (sr_health `ostr')  , end
initial:       log likelihood =   -26616.3
rescale:       log likelihood =   -26616.3
rescale eq:    log likelihood = -23689.019
Iteration 0:   log likelihood = -23689.019  (not concave)
Iteration 1:   log likelihood = -23608.069  (not concave)
Iteration 2:   log likelihood = -23606.768
  (output omitted)
Iteration 12:  log likelihood = -23598.033
Simultaneous bivariate ordered probit regression  Number of obs   =       9049
                                                  Wald chi2(23)   =     965.64
Log likelihood = -23598.033                       Prob > chi2     =     0.0000
------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
econrnk      |
     exp_ppl |   .1234239    .007774    15.88   0.000     .1081872    .1386605
       exp_2 |  -.4390104   .0373828   -11.74   0.000    -.5122794   -.3657414
       exp_3 |   .0323373   .0033307     9.71   0.000     .0258092    .0388654
   ind_nincm |   .2210446   .0339821     6.50   0.000     .1544409    .2876482
      hhsize |   .0572405   .0082824     6.91   0.000     .0410074    .0734736
      sskids |  -.0121174   .1149387    -0.11   0.916    -.2373932    .2131583
      sbkids |    .082298   .0823023     1.00   0.317    -.0790115    .2436076
       spens |  -.0664515   .0622542    -1.07   0.286    -.1884674    .0555644
     sawomen |   .2369426   .0752543     3.15   0.002     .0894469    .3844382
         age |  -.0507569   .0041248   -12.31   0.000    -.0588414   -.0426725
        age2 |   .0476939   .0043854    10.88   0.000     .0390987     .056289
    educat_2 |   .1107387   .0585926     1.89   0.059    -.0041007    .2255782
    educat_3 |    .055761   .0662793     0.84   0.400    -.0741441    .1856661
    educat_4 |   .1900912    .052825     3.60   0.000     .0865561    .2936264
    educat_5 |   .1917961   .0537008     3.57   0.000     .0865444    .2970478
    educat_6 |   .1812265   .0556275     3.26   0.001     .0721987    .2902544
     married |   .1790618   .0418981     4.27   0.000      .096943    .2611805
    divorced |  -.0952741   .0547488    -1.74   0.082    -.2025798    .0120317
   ltogether |  -.0270486   .0495721    -0.55   0.585    -.1242081    .0701108
     widowed |  -.0744711   .0560572    -1.33   0.184    -.1843412    .0353991
      hasjob |  -.0076203   .0315105    -0.24   0.809    -.0693797     .054139
      unem_b |  -.1803119   .0584877    -3.08   0.002    -.2949458   -.0656781
 mean_logexp |  -.1892784   .0428467    -4.42   0.000    -.2732564   -.1053005
-------------+----------------------------------------------------------------
sr_health    |
     exp_ppl |  -.0935869   .0134526    -6.96   0.000    -.1199534   -.0672203
       exp_2 |   .3525307    .053468     6.59   0.000     .2477353     .457326
       exp_3 |    -.02647   .0043955    -6.02   0.000     -.035085   -.0178551
   ind_nincm |  -.1202432   .0442456    -2.72   0.007    -.2069631   -.0335234
      hhsize |  -.0060836   .0154725    -0.39   0.694    -.0364092    .0242421
      sskids |   .0650188    .121711     0.53   0.593    -.1735304     .303568
      sbkids |  -.1308044   .0868541    -1.51   0.132    -.3010352    .0394265
       spens |   .0574917   .0655659     0.88   0.381    -.0710151    .1859985
     sawomen |  -.3625286   .0796968    -4.55   0.000    -.5187315   -.2063257
         age |    .023076   .0089286     2.58   0.010     .0055763    .0405758
        age2 |   -.042675   .0057854    -7.38   0.000    -.0540142   -.0313358
    educat_2 |   .0695105   .0705092     0.99   0.324    -.0686849     .207706
    educat_3 |   .1054092   .0746369     1.41   0.158    -.0408764    .2516948
    educat_4 |   .0525468   .0736925     0.71   0.476    -.0918878    .1969815
    educat_5 |   .0663105   .0760016     0.87   0.383    -.0826499    .2152709
    educat_6 |   .1217773   .0817214     1.49   0.136    -.0383937    .2819483
     married |  -.1797039   .0458573    -3.92   0.000    -.2695826   -.0898251
    divorced |  -.0243082   .0639908    -0.38   0.704    -.1497277    .1011114
```

```
   ltogether |   -.064558   .0546643    -1.18   0.238    -.1716981    .0425821
     widowed |  -.0227709   .0624224    -0.36   0.715    -.1451166    .0995747
      hasjob |   .1469339   .0405737     3.62   0.000      .067411    .2264569
      unem_b |   .2628048   .0616775     4.26   0.000     .1419191    .3836906
-------------+----------------------------------------------------------------
athrho       |
       _cons |  -.9656639   .2386267    -4.05   0.000    -1.433364   -.4979641
-------------+----------------------------------------------------------------
gamma        |
       _cons |   .8369609   .0895959     9.34   0.000     .6613562    1.012566
-------------+----------------------------------------------------------------
      /cut11 |  -2.159789   .1102107                     -2.375798    -1.94378
      /cut12 |  -1.455243   .1091092                     -1.669093   -1.241393
      /cut13 |  -.6745846   .1086014                     -.8874394   -.4617297
      /cut14 |   .0163376   .1084458                     -.1962124    .2288875
      /cut15 |   .8873047    .108811                       .674039     1.10057
      /cut16 |   1.335976   .1097295                       1.12091    1.551041
      /cut21 |  -2.328086   .4103713                     -3.132399   -1.523773
      /cut22 |  -1.507293   .2684242                     -2.033395   -.9811915
      /cut23 |  -.1557551   .0810297                     -.3145705    .0030603
      /cut24 |   1.061703   .2169265                      .6365349    1.486871
-------------+----------------------------------------------------------------
         rho |  -.7467926   .1055448                     -.8923539   -.4605145
------------------------------------------------------------------------------
LR test of indep. eqns. :          chi2(1) =   145.02   Prob > chi2 = 0.0000
------------------------------------------------------------------------------
```

The Likelihood Ratio or Wald test can be performed to test the independence of equations hypothesis ($\rho = 0$). By default, results of LR test are shown, but if either `claster` option or `pweight` was used, the Wald test that $\rho = 0$ will be reported. In case of SUR model, the alternative specification is to fit an univariate ordered probit model for each equation. For the simultaneous ordered probit model, the two-step estimator used above serves as an alternative. For our example, the null hypothesis is strongly rejected.

After estimating the model, we can get predicted probabilities for any combination of states. Lets calculate what is the predicted marginal probability of that average person in our sample reports his or her health as very bad ($SAH_i = 5$), and by how much this probability would change if percapita incomes and expenditures were to increase by 10%.

```
. predict p1, outcome(., #1)

. summarize p1
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          p1 |      9061    .0277679    .0519662   .0000298   .4838202

. local p1=r(mean)

. replace exp_ppl_11=exp_ppl_11*1.1
(9126 real changes made)

. replace exp_2=exp_2*(1.1)^2
(9126 real changes made)
```

```
. replace exp_3=exp_3*(1.1)^3
(9126 real changes made)

. replace ind_nincm=ind_nincm*1.1
(9273 real changes made)

. predict p2, outcome(.,#1)
. summarize p2

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          p2 |      9061    .0275784    .0517408   .0000184   .4818855
. display (r(mean)/`p1'-1)*100
-.68244268
```

The average effect of 10% increase in percapita incomes and expenditures would be the 0.7% reduction in predicted marginal probability of having bad health.

7 References

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Appendix

Here we derive analytical first and second derivatives of the likelihood function. As described in section 4, we define additional parameters: $r = arctanh(\rho)$; $d_{11} = c_{11}, d_{12} = \sqrt{c_{12} - c_{11}}, \dots, d_{1J-1} = \sqrt{c_{1J-1} - c_{1J-2}})$ and $d_{21} = c_{21}, d_{22} = \sqrt{c_{22} - c_{21}}, \dots, d_{2K-1} =$

$\sqrt{c_{2K-1} - c_{2K-2}}$). Therefore, complete set of parameters we estimate is

$$\Theta \equiv \{\beta_1, \beta_2, \gamma, r, d_{11}, \ldots, d_{1J-1}, d_{21}, \ldots, d_{2K-1}\}$$

First Derivatives

Then first derivative with respect to parameter $\theta \in \Theta$ is

$$\frac{\partial \ln L}{\partial \theta} = \frac{1}{L}\left(\frac{\partial \Phi_2(A_{11}, A_{21}, \rho')}{\partial \theta} - \frac{\partial \Phi_2(A_{12}, A_{22}, \rho')}{\partial \theta} - \frac{\partial \Phi_2(A_{13}, A_{23}, \rho')}{\partial \theta} + \frac{\partial \Phi_2(A_{14}, A_{24}, \rho')}{\partial \theta}\right)$$

lets introduce some more notation:

$$\begin{aligned}
A_{11} = A_{13} &= c_{1j} \quad - \beta_1' \mathbf{x}_{1i} \\
A_{12} = A_{14} &= c_{1j-1} - \beta_1' \mathbf{x}_{1i} \\
A_{21} = A_{22} &= \zeta(c_{2k} \quad - \gamma \beta_1' \mathbf{x}_{1i} - \beta_2' \mathbf{x}_{2i}) \\
A_{23} = A_{24} &= \zeta(c_{2k-1} - \gamma \beta_1' \mathbf{x}_{1i} - \beta_2' \mathbf{x}_{2i})
\end{aligned}$$

$\Phi^s \equiv \Phi_2(A_{1s}, A_{2s}, \tilde{\rho})$ for $s = 1 \ldots 4$

First partial derivatives of the bivariate standard normal CDF are:

$$\begin{aligned}
\Phi_1^s &= \phi(A_{1s}) F\left(\frac{A_{2s} - \tilde{\rho} A_{1s}}{\sqrt{1-\tilde{\rho}^2}}\right) \\
\Phi_2^s &= \phi(A_{2s}) F\left(\frac{A_{1s} - \tilde{\rho} A_{2s}}{\sqrt{1-\tilde{\rho}^2}}\right) \\
\Phi_3^s &= \frac{1}{2\pi\sqrt{1-\tilde{\rho}^2}} \exp^{-\frac{1}{2}\left(\frac{A_{1s}^2 + A_{2s}^2 - 2\tilde{\rho} A_{1s} A_{2s}}{1-\tilde{\rho}^2}\right)}
\end{aligned}$$

Then derivatives of A_{1s}, A_{2s} and $\tilde{\rho}$ with respect to each parameter are:

$$\begin{aligned}
\frac{\partial A_{1s}}{\partial \beta_1} &= -1 \\
\frac{\partial A_{1s}}{\partial d_{11}} &= 1 \\
\frac{\partial A_{1s}}{\partial d_{1j}} &= 2 d_{1j} \\
\frac{\partial A_{2s}}{\partial \beta_1} &= -\zeta \gamma \\
\frac{\partial A_{2s}}{\partial \beta_2} &= -\zeta \\
\frac{\partial A_{2s}}{\partial r} &= -A_{2s} \zeta^2 \gamma \frac{d\rho}{dr}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial A_{2s}}{\partial \gamma} &= -A_{2s}\zeta\tilde{\rho} - \zeta\beta_1 \\
\frac{\partial A_{2s}}{\partial d_{21}} &= \zeta \\
\frac{\partial A_{2s}}{\partial d_{2k}} &= 2\zeta d_{2k} \\
\frac{\partial \tilde{\rho}}{\partial r} &= \zeta(1-\zeta\gamma\tilde{\rho})\frac{d\rho}{dr} \\
\frac{\partial \tilde{\rho}}{\partial \gamma} &= \zeta(1-\tilde{\rho}^2) \\
\frac{d\rho}{dr} &= 4\frac{\exp^{2r}}{(1+\exp^{2r})^2}
\end{aligned}
$$

Second Derivatives

$$
\begin{aligned}
\frac{\partial^2 \ln L}{\partial\theta_1\partial\theta_2} &= L^{-1}\frac{\partial^2 L}{\partial\theta_1\partial\theta_2} - L^{-2}\frac{\partial L}{\partial\theta_1}\frac{\partial L}{\partial\theta_2} \\
\frac{\partial^2 L}{\partial\theta_1\partial\theta_2} &= \frac{\partial^2\Phi^1}{\partial\theta_1\partial\theta_2} - \frac{\partial^2\Phi^2}{\partial\theta_1\partial\theta_2} - \frac{\partial^2\Phi^3}{\partial\theta_1\partial\theta_2} + \frac{\partial^2\Phi^4}{\partial\theta_1\partial\theta_2}
\end{aligned}
$$

$$
\frac{\partial^2\Phi}{\partial\theta_1\partial\theta_2} = \sum_{ij=1}^{3}\Phi_{ij}^s\frac{\partial A_{is}}{\partial\theta_1}\frac{\partial A_{js}}{\partial\theta_2} + \Phi_1^s\frac{\partial^2 A_{1s}}{\partial\theta_1\partial\theta_2} + \Phi_2^s\frac{\partial^2 A_{2s}}{\partial\theta_1\partial\theta_2} + \Phi_3^s\frac{\partial^2 A_{3s}}{\partial\theta_1\partial\theta_2}
$$

where we used notation $A_{3s} \equiv \rho'$.

Second partial derivatives of the bivariate standard normal CDF are:

$$
\begin{aligned}
\Phi_{11}^s &= -A_{1s}\Phi_1^s - \tilde{\rho}\Phi_3^s \\
\Phi_{12}^s &= \Phi_3^s \\
\Phi_{13}^s &= \Phi_3^s\frac{\tilde{\rho}A_{2s} - A_{1s}}{1-\tilde{\rho}^2} \\
\Phi_{22}^s &= -A_{2s}\Phi_2^s - \tilde{\rho}\Phi_3^s \\
\Phi_{23}^s &= \Phi_3^s\frac{\tilde{\rho}A_{1s} - A_{2s}}{1-\tilde{\rho}^2} \\
\Phi_{33}^s &= \Phi_3^s\frac{A_{1s}A_{2s} + \rho' - \tilde{\rho}\frac{A_{1s}^2 + A_{2s}^2 - 2\tilde{\rho}A_{1s}A_{2s}}{1-\tilde{\rho}^2}}{1-\tilde{\rho}^2}
\end{aligned}
$$

$$
\frac{\partial^2 A_{1s}}{\partial d_{1j}^2} = 2
$$

$$
\begin{aligned}
\frac{\partial^2 A_{2s}}{\partial \beta_1 \partial r} &= \zeta^3 \gamma^2 \frac{d\rho}{dr} \\
\frac{\partial^2 A_{2s}}{\partial \beta_1 \partial \gamma} &= \zeta^2 \gamma \tilde{\rho} - \zeta \\
\frac{\partial^2 A_{2s}}{\partial \beta_2 \partial r} &= \zeta^3 \gamma \frac{d\rho}{dr} \\
\frac{\partial^2 A_{2s}}{\partial \beta_2 \partial \gamma} &= \zeta^2 \tilde{\rho} \\
\frac{\partial^2 A_{2s}}{\partial r^2} &= 3A_{2s} \zeta^4 \gamma^2 \left(\frac{d\rho}{dr} \right)^2 - A_{2s} \zeta^2 \gamma \frac{d^2 \rho}{dr^2} \\
\frac{\partial^2 A_{2s}}{\partial r \partial \gamma} &= \left(-3A_{2s} \zeta^2 (1\tilde{\rho}^2) + \beta_1 \zeta^3 \gamma \right) \frac{d\rho}{dr} \\
\frac{\partial^2 A_{2s}}{\partial r \partial d_{21}} &= -\zeta^3 \gamma \frac{d\rho}{dr} \\
\frac{\partial^2 A_{2s}}{\partial r \partial d_{2k}} &= -2d_{2k} \zeta^3 \gamma \frac{d\rho}{dr} \\
\frac{\partial^2 A_{2s}}{\partial \gamma^2} &= A_{2s} (3\zeta^3 \gamma \tilde{\rho} - \zeta^2) + 2\beta_1 \zeta^2 \tilde{\rho} \\
\frac{\partial^2 A_{2s}}{\partial \gamma \partial d_{21}} &= -\zeta^2 \tilde{\rho} \\
\frac{\partial^2 A_{2s}}{\partial \gamma \partial d_{2k}} &= -2d_{2k} \zeta^2 \tilde{\rho} \\
\frac{\partial^2 A_{2s}}{\partial d_{2k}^2} &= 2
\end{aligned}
$$